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ON CHARACTER PSEUDO - AMENABLE SEMIGROUP ALGEBRAS.

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Abstract. We study the character pseudo - amenability of semigroup algebras. We focus on certain semigroups such as inverse semigroup with uniformly locally finite idempotent set and Brandt semigroup and study the character pseudo - amenability of semigroup algebra $l^1(S)$ in relation to the semigroup S. In particular, we show that for a unital cancellative semigroup S, the character pseudo-amenability of $l^1(S)$ is equivalent to its amenability, this is in turn equivalent to S being an amenable group.

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1. Introduction

The notion of amenability in Banach algebra was initiated by Johnson in [13]. Since then, amenability has become a major issue in Banach algebra theory and in harmonic analysis. For details on amenability in Banach algebras see [17].

In [8], Ghahramani and Loy introduced generalized notions of amenability with the hope that it will yield Banach algebra without bounded approximate identity which nonetheless had a form of amenability. All known approximate amenable Banach algebras have bounded approximate identities until recently when Ghahramani and Read in [10] give examples of Banach algebras which are boundedly approximately amenable but which do not have bounded approximate identities. This answers a question open since the year 2004 when Ghahramani and Loy founded the notion of approximate amenability.

Let A be a Banach algebra over $\mathbb{C}$ and $\varphi : A \to \mathbb{C}$ be a character on A, that is, an algebra homomorphism from A into $\mathbb{C}$, and let Φ_A denote the character space of A (that is, the set of all characters on A). In [23], Monfared introduced the notion of character amenable Banach algebras, see also [6] for this notion. His definition of this notion requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are concerned where either the left or right module action is

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defined by characters on A. As such character amenability is weaker than the classical amenability introduced by Johnson in [13], so all amenable Banach algebras are character amenable. For details on character amenability in Banach algebras see [18].

In [24], the authors introduced and studied a notion of character amenability based on the existence of a φ-approximate diagonal that is not necessarily bounded. They characterized this notion in terms of derivations from a Banach algebra A into certain Banach A-bimodules.

Various aspects of cohomologies of semigroup algebras have been studied by several authors, most notably are Duncan and Namioka [4], Sadr and Pourabbas [26], Lashkarizadeh and Samea [1], Grønbaek [7], Mewomo [16], Mewomo and Ogunsola [20], [21], Mewomo and Maepa [19] and Dales, Lau and Strauss [3].

It is shown in [4] that the amenability of semigroup algebra $l^1(S)$ implies that the semigroup S is amenable. The authors in [26] characterized the approximate amenability of Brandt semigroup algebras. In particular, they showed that for a Brandt semigroup S over a group G with nonempty index set J, the semigroup algebra $l^1(S)$ is approximately amenable if and only if G is amenable and J is finite. The authors in [19] studied the character amenability of $l^1(S)$ in relation to the structures of the semigroup S. In particular, they showed that for any semigroup S, if $l^1(S)$ is character amenable, then S is regular and amenable.

In this work, we study the character pseudo - amenability of semigroup algebras. We focus on certain semigroups such as inverse semigroup with uniformly locally finite idempotent set and Brandt semigroup and study the character pseudo - amenability of $l^1(S)$ in relation to the semigroup S.

2. Preliminaries

First, we recall some standard notions; for further details, see [2] and [17].

Let A be an algebra. The character space of A is denoted by Φ_A. Let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A).$$

For example, for $x \in X$, the map $\delta_x : A \to X$ defined by $\delta_x(a) = a \cdot x - x \cdot a \quad (a \in A)$ is a derivation; derivations of this form are called the *inner derivations.*

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k \|a\| \|x\|, \quad \|x \cdot a\| \leq k \|a\| \|x\| \quad (a \in A, x \in X).$$

By renorming X, we can suppose that $k = 1$. For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations specified by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle, \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$

is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A,X) = \mathcal{Z}^1(A,X)/\mathcal{N}^1(A,X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A,X') = \{0\}$ for each Banach A-bimodule X.

A derivation $D : A \to X$ is *approximately inner* if there is a net (x_v) in X such that

$$D(a) = \lim_v (a \cdot x_v - x_v \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|.\|)$. That is, $D(a) = \lim_v \delta_{x_v}(a)$, where (δ_{x_v}) is a net of inner derivations. The Banach algebra A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is approximately inner.

Given a Banach algebra A, and $\varphi \in \Phi_A$, we let $\mathcal{M}^A_{\varphi_r}$ denote the class of Banach A- bimodules X for which the right module action of A on X is given by $x \cdot a = \varphi(a)x \quad (a \in A, x \in X)$, and $\mathcal{M}^A_{\varphi_l}$ denote the class of Banach A- bimodules X for which the left module action of A on X is given by $a \cdot x = \varphi(a)x \quad (a \in A, x \in X)$. If the right module action of A on X is given by $x \cdot a = \varphi(a)x$, then it is easy to see that the left module action of A on the dual module X' is given by $a \cdot f = \varphi(a)f \quad (a \in A, f \in X')$. The converse also holds, and so we have that $X \in \mathcal{M}^A_{\varphi_r}$ (resp. $X \in \mathcal{M}^A_{\varphi_l}$) if and only if $X' \in \mathcal{M}^A_{\varphi_l}$ (resp. $X' \in \mathcal{M}^A_{\varphi_r}$).

Let A be a Banach algebra and let $\varphi \in \Phi_A$, we recall the following definitions from [12], [23]:
(i) A is left φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_r}$;
(ii) A is right φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_l}$;
(iii) A is left character amenable if it is left φ-amenable for every $\varphi \in \Phi_A$;
(iv) A is right character amenable if it is right φ-amenable for every $\varphi \in \Phi_A$;
(v) A is character amenable if it is both left and right character amenable.

We also recall the following definitions from [12] that, for $\varphi \in \Phi_A$, a left (right) φ-approximate diagonal for A is a net (m_α) in $A\hat{\otimes}A$ such that

$$(i)\ \|m_\alpha \cdot a - \varphi(a)m_\alpha\| \to 0 \quad (\|a \cdot m_\alpha - \varphi(a)m_\alpha\| \to 0) \quad (a \in A);$$

$$(ii)\ \langle \varphi \otimes \varphi, m_\alpha \rangle = \varphi(\pi(m_\alpha)) \to 1,$$

where $\pi : A\hat{\otimes}A \to A$ defined by $\pi(a \otimes b) = ab \quad (a, b \in A)$ is the product map.

Let A be a Banach algebra and $\varphi \in \Phi_A$. We recall from [24] that
(i) A is left (right) φ-pseudo - amenable if it has a left (right) φ-approximate diagonal;
(i) A is left (right) character pseudo - amenable if it has a left (right) φ-approximate diagonal for every $\varphi \in \Phi_A$;
(iii) A is character pseudo - amenable if it is both left and right character pseudo - amenable.

3. General Results

In this section, we prove some general results which are useful in establishing our main results on semigroup algebras.

Proposition 3.1. *Let A and B be Banach algebras and let $\varphi \in \Phi_B$. Suppose there is a continuous homomorphism $\tau : A \to B$ with dense range.*
(i) If A is left (right) $\varphi \circ \tau$-amenable, then B is left (right) φ-amenable.
(ii) If A is left (right) character amenable, then B is left (right) character amenable.

Proof (i) This is Proposition 3.5 of [15]. We provide a proof different from that given in [15]. Suppose A is right $\varphi \circ \tau$-amenable. Let X be a Banach B-bimodule such that $b \cdot x = \varphi(b)x \quad (b \in B, x \in X)$. Let $Y := X$ be the Banach A-bimodule with actions given by $a \cdot x = \tau(a) \cdot x \quad (a \in A, x \in X)$. If $D : B \to X'$ is a continuous derivation, then $D \circ \tau : A \to Y'$ is a continuous derivation. Thus, there exists $f \in X'$ such that $(D \circ \tau)(a) = \delta_f(a) \quad (a \in A)$; that is, D is inner and so B is right φ-amenable. The proof of the left version is similar.
(ii) This follows from (i). □

Let I be a closed ideal of A, then there is a unique $\varphi_q \in \Phi_{A/I}$ with $\varphi_q \circ q = \varphi \quad (\varphi \in \Phi_A)$ if and only if $I \subseteq ker\varphi$ where $q : A \to A/I$ is the quotient map.

Corollary 3.2. *Let A be a left (right) φ-amenable Banach algebra with $\varphi \in \Phi_A$ and let I be a closed ideal of A. Then A/I is left (right) φ_q-amenable.*

Proof This follows from Proposition 3.1 (i). □

Let $A \hat{\otimes} B$ be the projective tensor product of Banach algebras A and B. For $f \in A', g \in B'$, let $f \otimes g \in (A \hat{\otimes} B)'$ such that

$$(f \otimes g)(a \otimes b) = f(a)g(b) \quad (a \in A, b \in B).$$

Then $\Phi_{A \hat{\otimes} B} = \{\varphi \otimes \psi : \varphi \in \Phi_A, \psi \in \Phi_B\}$.

Theorem 3.3. *Let A and B be Banach algebras.*
(i) Suppose $\varphi \in \Phi_A$, $\psi \in \Phi_B$ and $A \hat{\otimes} B$ is left (right) $\varphi \otimes \psi$-amenable. Then A is left (right) φ-amenable and B is left (right) ψ-amenable.
(ii) Suppose $A \hat{\otimes} B$ is left (right) character amenable. Then A is left (right) character amenable and B is left (right) character amenable.

Proof (i) This is Theorem 3.3 of [15]. We provide a proof different from that given in [15]. Clearly the maps $\tau_1 : A \hat{\otimes} B \to A$ and $\tau_2 : A \hat{\otimes} B \to B$ defined by $\tau_1(a \otimes b) = \psi(b)a$ and $\tau_2(a \otimes b) = \varphi(a)b \quad (a \in A, b \in B)$ determine continuous homomorphisms with dense range. Thus the result follows from Proposition 3.1(i)
(ii) This follows from (i) □

We next give some general results on character pseudo - amenability.

Proposition 3.4. *Let A and B be Banach algebras. Suppose there exists a continuous epimorphism τ from A onto B.*
(i) If A is left (right) $\varphi \circ \tau$-pseudo - amenable, then B is left (right) φ-pseudo - amenable for $\varphi \in \Phi_B$;

(ii) If A is left (right) character pseudo - amenable, then B is left (right) character pseudo - amenable.

Proof (i) Since A is left (right) $\varphi\circ\tau$-pseudo - amenable, then it has a left (right) $\varphi\circ\tau$-approximate diagonal. Also, since $\tau : A \to B$ is a continuous epimorphism, then $\tau\otimes\tau : A\hat{\otimes}A \to B\hat{\otimes}B$ maps left (right) $\varphi\circ\tau$-approximate diagonal for A to a left (right) φ-approximate diagonal for B. Thus B is left (right) φ-pseudo - amenable.
(ii) This follows from (i) □

Corollary 3.5. *Let A be a left (right) φ- pseudo - amenable Banach algebra with $\varphi \in \Phi_A$ and let I be a closed ideal of A. Then A/I is left (right) φ_q-pseudo - amenable.*

Proof This follows from Proposition 3.4 (i). □

Proposition 3.6. *Let A and B be Banach algebras and $\varphi \in \Phi_A, \psi \in \Phi_B$. Suppose $A\hat{\otimes}B$ is left (right) $\varphi\otimes\psi$-pseudo amenable and B contains a non-zero idempotent. Then A is left (right) φ-pseudo - amenable.*

Proof This follows from a small modification of the proof of Proposition 3.5 in [14]. It is easy to see that if (m_α) is a left (right) $\varphi\otimes\psi$-approximate diagonal for $A\hat{\otimes}B$ and $\tau : (A\hat{\otimes}B)\hat{\otimes}(A\hat{\otimes}B) \to (A\hat{\otimes}A)$, then $(\tau(m_\alpha))$ is a left (right) φ-approximate diagonal for A. Thus A is left (right) φ-pseudo - amenable. □

Let A be a Banach algebra and let J be a non-empty set. We denote by $\mathcal{M}_J(A)$ the set of $J\times J$ matrices (a_{ij}) with entries in A such that

$$\|(a_{ij})\| = \sum_{i,j\in J} \|a_{ij}\| < \infty.$$

Then $\mathcal{M}_J(A)$ with the usual matrix multiplication is a Banach algebra that belongs to the class of l^1-Munn algebras introduced in [5]. The map $\tau : \mathcal{M}_J(A) \to A\hat{\otimes}\mathcal{M}_J(\mathbb{C})$ defined by $\tau((a_{ij})) = \sum_{i,j\in J} a_{ij}\otimes E_{ij}$ $((a_{ij}) \in \mathcal{M}_J(A))$, is an isometric isomorphism of Banach algebras, where (E_{ij}) are the matrix units in $\mathcal{M}_J(\mathbb{C})$. With this, we have the next result.

Corollary 3.7. *Let A be a Banach algebra and J a non-empty set. If $\mathcal{M}_J(A)$ is left (right) character pseudo - amenable, then A is left (right) character pseudo - amenable.*

Proof Since $\mathcal{M}_J(A) \cong A\hat{\otimes}\mathcal{M}_J(\mathbb{C})$. The result follows from Proposition 3.6 and the fact that $\mathcal{M}_J(\mathbb{C})$ contains many non-zero idempotents. □

We recall from [11] that a Banach algebra A is pseudo - amenable if there is a net $(u_\alpha) \subset A\hat{\otimes}A$, called an approximate diagonal for A, such that $au_\alpha - u_\alpha a \to 0$ and $\pi(u_\alpha)a \to a$ for each $a \in A$.

Theorem 3.8. *Let A be a Banach algebra. Suppose A is pseudo - amenable with approximate diagonal (u_α). Let X be a Banach A-bimodule such that for each $\lambda \in X'$, there exists α_0 with $\lambda\cdot\pi(u_\alpha) = \lambda$, for all $\alpha \geq \alpha_0$. Then every derivation from A into X' is approximately inner.*

Proof Let $u_\alpha = \sum_i a_i^{(\alpha)} \otimes b_i^{(\alpha)}$ such that $\sum_i \|a_i^{(\alpha)}\|\|b_i^{(\alpha)}\| < \infty$ for each α, and let $D : A \to X'$ be a derivation. Since (u_α) is an approximate diagonal for A, we have $a \cdot u_\alpha - u_\alpha \cdot a \to 0$, that is

$$\sum_i a \cdot a_i^{(\alpha)} \otimes b_i^{(\alpha)} - \sum_i a_i^{(\alpha)} \otimes b_i^{(\alpha)} \cdot a \to 0 \quad (a \in A).$$

It follows that

$$\sum_i D(aa_i^{(\alpha)}) \cdot b_i^{(\alpha)} - \sum_i D(a_i^{(\alpha)}) \cdot b_i^{(\alpha)} a \to 0$$

and so,

$$D(a) \cdot \sum_i a_i^{(\alpha)} \cdot b_i^{(\alpha)} + a \cdot \sum_i D(a_i^{(\alpha)}) \cdot b_i^{(\alpha)} - \sum_i D(a_i^{(\alpha)}) \cdot b_i^{(\alpha)} a \to 0 \quad (a \in A).$$

That is,

$$D(a) \cdot \pi(u_\alpha) + a \cdot \sum_i D(a_i^{(\alpha)}) \cdot b_i^{(\alpha)} - \sum_i D(a_i^{(\alpha)}) \cdot b_i^{(\alpha)} a \to 0 \quad (a \in A)$$

which gives

$$D(a) \cdot \pi(u_\alpha) - a \cdot \lambda_\alpha + \lambda_\alpha \cdot a \to 0 \quad (a \in A) \tag{3.1}$$

where $\lambda_\alpha = -\sum_i D(a_i^{(\alpha)}) \cdot b_i^{(\alpha)} \in X'$. Using our assumption, for each $a \in A$, there exists α_0 such that
$D(a) \cdot \pi(u_\alpha) = D(a)$, for all $\alpha \geq \alpha_0 \quad (a \in A)$. Thus by (3.1), we have

$$D(a) = \lim_\alpha a \cdot \lambda_\alpha - \lambda_\alpha \cdot a \quad (a \in A)$$

and so D is approximately inner. □

The following result is Theorem 2.2 of [24]. The left version follows from the fact that any statement about right character pseudo -amenability turns into an analogues statement about left character pseudo - amenability (and vice versa) by simply replacing A by its opposite algebra.

Proposition 3.9. *Let A be a Banach algebra and $\varphi \in \Phi_A$. Then the following statements are equivalent:*
(i) A is right φ-pseudo-amenable.
(ii) For each $X \in \mathcal{M}_{\varphi_l}^A$, any derivation $D : A \to X'$ is approximately inner.

By using Theorem 3.8 and Proposition 3.9, we have the next result.

Proposition 3.10. *Let A be a pseudo-amenable Banach algebra. Then A is left φ-pseudo-amenable for each $\varphi \in \Phi_A$.*

Proof Let $X \in \mathcal{M}_{\varphi_r}^A$ and let $D : A \to X'$ be a derivation. Since A is pseudo - amenable, it has an approximate diagonal $(u_\alpha) \subset A\hat{\otimes}A$. Also, using the fact that $(\pi(u_\alpha))$ is an approximate identity for A, we have $\varphi(\pi(u_\alpha)) \to 1$. Then by passing to a subnet, we may suppose that $\varphi(\pi(u_\alpha)) \neq 0$ and so, we define $m_\alpha = \frac{u_\alpha}{\varphi(\pi(u_\alpha))}$. Clearly (m_α) is an approximate diagonal for A and for each $\lambda \in X'$, we have

$$\lambda \cdot \pi(m_\alpha) = \varphi(\pi(m_\alpha))\lambda = \lambda.$$

Thus by Theorem 3.8, D is approximately inner and so, using Proposition 3.9, we conclude that A is left φ-pseudo-amenable. □

Remark: A in the above result can also be prove in a similar way to be right φ-pseudo - amenable. Thus Proposition 3.10 shows that pseudo - amenability implies character pseudo - amenability.

The final result in this section shows that character pseudo - amenability of the group algebra $L^1(G)$ is completely characterized by the amenability of the underlying group G. The character amenability, approximate amenability, amenability and pseudo - amenability versions of the result have been proved in [23], [8], [13] and [11] respectively. In view of Proposition 3.9, the proof for the character pseudo - amenability case carries over without difficulty.

Proposition 3.11. *For a locally compact group G. The following are equivalent:*
(i) $L^1(G)$ is left character pseudo - amenable.
(ii) $L^1(G)$ is right character pseudo - amenable.
(iii) G is amenable.

4. Results on semigroup algebras

In this section, we shall consider the character pseudo - amenability properties of semigroup algebras. We briefly recall the following definitions and notations. For details, see [3] and [25].

Let S be a semigroup, S is said to be regular if for each $s \in S$, there is $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. S is an inverse semigroup if such s^* exists and is unique for each $s \in S$. We shall denote the inverse of an element s in an inverse semigroup S by s^{-1}.

For each $s \in S$, define $L_s(t) = st, R_s(t) = ts \quad (t \in S)$. An element $s \in S$ is left (resp. right) cancellable if L_s (resp. R_s) is injective on S, and s is cancellable if it is both left cancellable and right cancellable. The semigroup S is left (resp. right) cancellative if each element in S is left (resp. right) cancellable, and cancellative if each element is cancellable.

Let S be a semigroup, we shall use the following notations and definitions from [25]. For $s, t \in S$, we define the sets

$$[st^{-1}] = \{u \in S : ut = s\}, [t^{-1}s] = \{u \in S : tu = s\}.$$

For $s, t \in S$, we define a relation $\mathbb{D}$ on S by $s\mathbb{D}t$ if and only if there exists $x \in S$ with $Ss \bigcup\{s\} = Sx \bigcup\{x\}$ and $xS \bigcup\{x\} = tS \bigcup\{t\}$. This is an equivalence relation, see [25].

An element $p \in S$ is idempotent if $p^2 = p$. The set of idempotents in S is denoted by $E(S)$. A semigroup S is semilattice if S is commutative and $E(S) = S$.

The following characterization of $\mathbb{D}-$ classes in an inverse semigroup is Proposition 2.11 of [25].

Proposition 4.1. *Let S be an inverse semigroup, and let $s, t \in S$. Then $s\mathbb{D}t$ if and only if there exists $x \in S$ with $s^{-1}s = xx^{-1}$ and $t^{-1}t = x^{-1}x$.*

Let S be an inverse semigroup, and $p \in E(S)$. We set

$$G_p = \{s \in S : ss^{-1} = s^{-1}s = p\},$$

where s^{-1} denote the inverse of s. Then G_p is a group with identity p and G_p contains any other subgroup of S with identity p. Thus G_p is called the maximal subgroup of S at p.

Let P be a partially ordered set. For $p \in P$, we define

$$(p] = \{x : x \leq p\} \text{ and } [p) = \{x : p \leq x\}.$$

Then P is locally finite if $(p]$ is finite for each $p \in P$, and is locally $C-$ finite for some constant $C > 1$ if $\mid (p] \mid < C$ for each $p \in P$. A partially ordered set that is $C-$ finite for some C is uniformly locally finite.

Let S be an inverse semigroup. Then S is [locally finite/ C-locally finite/ uniformly locally finite] if the partially ordered set $(E(S), \leq)$ has the corresponding property.

The following result is Proposition 2.14 of [25].

Proposition 4.2. *Let S be an inverse semigroup. Suppose that $(E(S), \leq)$ is [uniformly] locally finite. Then $(S, \leq)$ is [uniformly] locally finite.*

Let $\{A_\lambda : \lambda \in J\}$ be a collection of Banach algebras. Then the l^1- direct sum $l^1 - \bigoplus\{A_\lambda : \lambda \in J\}$ is a Banach algebra with respect to component-wise multiplication, where multiplication in the λ^{th} component is just the multiplication in A_λ.

Let $\{D_\lambda : \lambda \in J\}$ be the collection of all $\mathbb{D}$-classes of S, $E(D_\lambda) = E(S) \bigcap D_\lambda$. The maximal subgroup of S at $p_\lambda \in E(D_\lambda)$ is denoted by G_{p_λ}.

The following theorem from [25] gives the structure of semigroup algebra $l^1(S)$ for uniformly locally finite inverse semigroup S and is useful in our main result of this section.

Theorem 4.3. *Let S be a uniformly locally finite inverse semigroup with $\mathbb{D}-$ classes $\{D_\lambda : \lambda \in J\}$. For each λ take an idempotent $p_\lambda \in D_\lambda$. Then there is an isomorphism of Banach algebras*

$$l^1(S) \cong l^1 - \bigoplus\{\mathbb{M}_{E(D_\lambda)}(l^1(G_{p_\lambda})) : \lambda \in J\}.$$

We recall that a discrete semigroup S is left amenable if the space $l^\infty(S)$ admits a functional m called a mean such that $m(1) = 1 = \|m\|$ and the mean is left invariant, i.e. $m(l_x f) = m(f)$, where $(l_x f)(y) = f(xy) \quad (x, y \in S, f \in l^\infty(S))$. Similarly for right amenable. If S is both left and right amenable, it is amenable. In the case of a group, or even an inverse semigroup, left (or right) amenability implies amenability.

In the next result, we show that if S is a unital left (or right) cancellative discrete semigroup and $l^1(S)$ is left (or right) character pseudo - amenable, then S is an amenable group.

Theorem 4.4. *Let S be a unital left (or right) cancellative semigroup. Suppose $l^1(S)$ is left (or right) character pseudo - amenable, then S is an amenable group.*

Proof We prove the left version, the right version follows similar argument.We follow the argument of [1, Thorem 1.1], $l^\infty(S)$ is a Banach $l^1(S)$-bimodule. If we let $X = l^\infty(S)/\mathbb{C}1$ such that $X \in \mathcal{M}^A_{\varphi_r}$. Since $l^1(S)$ is left character pseudo - amenable, then by Proposition 3.9, every derivation

$D : l^1(S) \to X'$ is approximately inner, and so there exists a net (λ_α) in X' such that

$$D(f) = \lim_\alpha (f \cdot \lambda_\alpha - \lambda_\alpha \cdot f) \quad (f \in l^1(S)).$$

Similar argument as in the later end of [1, Theorem 1.1] now gives a left invariant mean on $l^\infty(S)$, so that S is left amenable semigroup. Since S is unital, it has an identity e_S such that $e_S \cdot s = s \cdot e_S = s \quad (s \in S)$. It is easy to see that S is a regular semigroup. Thus for each $s \in S$, there exists $s^* \in S$ such that $ss^*s = s = se_S$ and $s^*ss^* = s^* = s^*e_S$. Also, since S is left cancellative, we have $s^*s = ss^* = e_S$ and so S is a group. Hence S is an amenable group. □

The next result is from [7, Theorem 2.3].

Theorem 4.5. *Let S be a unital and left or right cancellative semigroup. Then $l^1(S)$ is amenable if and only if S is an amenable group.*

Corollary 4.6. *Let S be a unital cancellative semigroup. Then the following are equivalent:*
(i) $l^1(S)$ is character pseudo - amenable.
(ii) S is an amenable group.
(iii) $l^1(S)$ is amenable.

Proof (i) ⇒ (ii) follows from Theorem 4.4.
(ii) ⇒ (iii) follows from Theorem 4.5.
(iii) ⇒ (i) is clear. □

Theorem 4.7. *Let S be an inverse semigroup such that $(E(S), \leq)$ is uniformly locally finite. Suppose $l^1(S)$ is left or right character pseudo - amenable. Then each maximal subgroup of S is amenable.*

Proof By Proposition 4.2, $(S, \leq)$ is uniformly locally finite since $(E(S), \leq)$ is uniformly locally finite. Thus, using Theorem 4.3, we have

$$l^1(S) \cong l^1 - \bigoplus \{\mathbb{M}_{E(D_\lambda)}(l^1(G_{p_\lambda})) : \lambda \in J\},$$

and so, for each $\lambda \in J$, $\mathbb{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$ is a homomorphic image of $l^1(S)$. Thus, by Proposition 3.4, we have

$$\mathbb{M}_{E(D_\lambda)}(l^1(G_{p_\lambda})) \cong \mathbb{M}_{E(D_\lambda)}(\mathbb{C}) \hat{\otimes} (l^1(G_{p_\lambda}))$$

is left character pseudo - amenable for each $\lambda \in J$. Also, $l^1(G_{p_\lambda})$ is left character amenable by Corollary 3.7, and so G_{p_λ} is an amenable group by Proposition 3.11. □

We recall that a Brandt semigroup S over a group G with index set J consists of all canonical $J \times J$ matrix units over $G \bigcup \{0\}$ and a zero matrix 0. It is an inverse semigroup over G with index set J given by

$$S = \{(g)_{ij} : g \in G, i, j \in J\} \bigcup \{0\},$$

where $(g)_{ij}$ is the $J \times J$-matrix with (k, l)-entry equal to g if $(k, l) = (i, j)$ and 0 if $(k, l) \neq (i, j)$ and multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if} \quad j = k, \\ \mathbf{0} & \text{if} \quad j \neq k. \end{cases}$$

It was shown in [4] that for a Brandt semigroup S over a group G with a finite index set J, $l^1(S)$ is amenable if and only if G is amenable. It was also shown in [1] that $l^1(S)$ is approximately amenable if and only if G is amenable.

Corollary 4.8. *Let S be a Brandt semigroup over a group G with index set J. Suppose $l^1(S)$ is left or right character pseudo - amenable. Then G is amenable.*

Proof It is easy to see that $E(S) = \{(e_G)_{ii} : i \in J\} \bigcup \{0\}$. Also, since $E(S)$ is semilattice, we have

$$(x] = \{y \in E(S) : y = yx\} = E(S)x = \begin{cases} \{0\} & \text{if} \quad x = 0, \\ \{0, x\} & \text{if} \quad x \neq 0, \end{cases}$$

for each $x \in E(S)$. Thus, $(E(S), \leq)$ is uniformly locally finite, and so the result follows by using Theorem 4.7. □

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